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The Scheme of 10th Order Implicit Runge-Kutta Method to Solve the First Order of Initial Value Problems

Z. Bahri[1], L. Zakaria[2,*], Syamsudhuha[1]

Abstract—To construct a scheme of implicit Runge-Kutta methods, there are a number of coefficients that must be determined and satisfying consistency properties and Butcher's simplifying assumptions. In this paper we provide the numerical simulation technique to obtain a scheme of 10th order Implicit Runge-Kutta (IRK_{10}) method. For simulation process, we construct an algorithm to compute all the coefficients involved in the IRK_{10} scheme. The algorithm is implemented in a language programming (Turbo Pascal) to obtain all the required coefficients in the scheme. To show that our scheme works correctly, we use the scheme to solve Hénon-Heiles system.

Keywords—ODEs, 10th order IRK method, numerical technique, Hénon-Heiles system

I. Introduction

Let a first order ordinary differential equation system (ODES)

$$y' = f\left(x, y\left(x\right)\right) \quad (1)$$

together with

$$y(x_0) = y_0 \quad (2)$$

In (1), where the "prime"indicates differentiation with respect to *x*, *y* is a *D*-dimensional vector $\left(y \in \Re^D\right)$, and $f : \Re x \Re^D \rightarrow \Re^D$. The ODES (1)-(2) is the well-known the first order of initial value problem. To solve problem (1)-(2) can be used analytical and/or numerical procedures. But, for solving the special problems (*Hamiltonian and Divergen Free systems*, for examples) and taking efficiency and effective calculations, some mathematicians recommended to use numerical approximations ([1-4,5]). One of numerical methods which can be used to solve (1-2) which is enough recognized and a lot of used is Runge-Kutta method.

Definition : Let $b_i, a_{ij}, and\ c_i\left(i, j = 1, 2, \cdots, s\right)$ be real numbers. The method

$$y_{n+1} = y_n + {}^{2}\sum_{i=1}^{s} b_i \quad f_i \quad (3)$$

$$f_i = f\left(t_n + c_i{}^{2}\ , y_n + {}^{2}\sum_{j=1}^{s} a_{ij} \quad f_j\right)$$

is called an *s-stage* Runge-Kutta method. When $a_{ij} = 0$ for $i \le j$ the method is called an **explicit Runge-Kutta** method. If $a_{ij} = 0$ $i < j$ and least one $a_{ii} \neq 0$, the method is called a

diagonal implicit Runge-Kutta method. If all of the diagonal elements are identical, $a_{ij} = \gamma$ for i = 1,2,...,*s*, the method is called a **singly diagonal implicit Runge-Kutta** method. In all other cases, the method is called an **implicit Runge-Kutta** method. The Runge-Kutta methods are often given in the form of a tableau containing their coefficients namely

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s \end{array} \quad (4)$$

where $c_i = \sum_{j=1}^{s} a_{ij}\ ; i = 1, 2, \cdots, s$.

Consistency in the Runge-Kutta methods is investigated by using a Taylor series expansion. Any *s*-stage Runge-Kutta process of the form (3) or (4) is consistent if $\sum_{i=1}^{s} b_i = 1$. Butcher in [6] states that this condition is necessary and sufficient condition for the local truncation error of the method to have *asymptotic behavior* $O\left({}^{2}\right)$.

There are some advantages to use Implicit Runge-Kutta (IRK) methods (see [6-8]). (1) IRK methods usually are required for systems whose solutions contain rapidly decaying components (see [6]); (2) Some methods may also be used in preserving the symplectic structure of Hamiltonian systems; (3) IRK methods (*Gauss quadratures*) have a number of big potency for the computing of integrate the geometry; (4) high order integrator to be used by the reason of doubled accuracy is recommended; (5) to evaluate the vector field "costly", all stage-*s* in IRK can be evaluated by parallel and; (6) IRK can be solve the ordinary differential equation (ODE) or system (ODEs) in general.

To construct a scheme of IRK methods, Butcher discovered the existence of *s*-stage methods of order 2*s*, for all *s*. He used *simplifying assumptions* to find these methods. The simplifying assumptions are

[1]Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Riau, Kampus Bina Widya Simpang Baru Km. 12.5 Pekanbaru, Riau, Indonesia.

[2]Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Lampung, Soemantri Brojonegoro 1 Bandar Lampung, Lampung, Indonesia.

[*]Correspondence to 2nd Author, email: lazakaria.1969@fmipa.unila.ac.id. Tel.:+62-721-704625; fax: +62-721-704625.

$$A(\,,\,)=\sum_{j=1}^{s}a_{ij}c_j^{q-1}=\frac{1}{q}c_i^q \qquad i=1,2,...,s \quad q=1,2,...,\,,$$

$$B(p)=\sum_{i=1}^{s}b_i c_i^{q-1}=\frac{1}{q} \qquad q=1,2,\dots,p \tag{5}$$

$$C(\,\circ\,)=\sum_{i=1}^{s}b_i c_i^{q-1}a_{ij}=\frac{b_j}{q}\left(1-c_j^q\right) \quad j=1,2,...,s \quad q=1,2,...,\circ$$

Based on consistency property and Butcher's simplifying assumptions, Hairer *et.al* (2007)[7] and Ismail (2009)[8] derive some IRK methods new analytical procedures.

In analytical procedure, we can use the idea of collocation to derive IRK methods ([4],[8]). Unfortunately, for the stage of *stage* $s \geq 3$, constructing an integrator IRK using analytical procedure will be difficult because there are some values of b_i, c_i, and a_{ij} $(i, j=1,2,...,s)$ must be determined. Therefore, the numerical technique or procedure is an alternative choice to use. In this procedure, we can use computer to determine the values of c_i $(i=1,2,...,s)$, b_i and a_{ij} $(i, j=1,2,...,s)$.

Notice that one important property of IRK method is symplecticness. An IRK method (4) is symplectic if it satisfy the following condition ([9])

$$b_i b_j - b_i a_{ij} - b_j a_{ji} = 0 \quad , \; i, j \in \{1,2,...,s\}.$$

II. Result and Discussion

In this section, we describe how to get a class of 10th order IRK method using computer simulation to obtain all Butcher's coefficient values (4) based on Butcher's simplifying assumptions (5).

Setting $k=i=j=s=1,2,3,4,5$ into first equation in (5), we have the following systems

$$a_{11}+a_{12}+a_{13}+a_{14}+a_{15}=c_1 \tag{6.1}$$
$$a_{21}+a_{22}+a_{23}+a_{24}+a_{25}=c_2 \tag{6.2}$$
$$a_{31}+a_{32}+a_{33}+a_{34}+a_{35}=c_3 \tag{6.3}$$
$$a_{41}+a_{42}+a_{43}+a_{44}+a_{45}=c_4 \tag{6.4}$$
$$a_{51}+a_{52}+a_{53}+a_{54}+a_{55}=c_5 \tag{6.5}$$
$$a_{11}c_1+a_{12}c_2+a_{13}c_3+a_{14}c_4+a_{15}c_5=\tfrac{1}{2}c_1^2 \tag{7.1}$$
$$a_{21}c_1+a_{22}c_2+a_{23}c_3+a_{24}c_4+a_{25}c_5=\tfrac{1}{2}c_2^2 \tag{7.2}$$
$$a_{31}c_1+a_{32}c_2+a_{33}c_3+a_{34}c_4+a_{35}c_5=\tfrac{1}{2}c_3^2 \tag{7.3}$$
$$a_{41}c_1+a_{42}c_2+a_{43}c_3+a_{44}c_4+a_{45}c_5=\tfrac{1}{2}c_4^2 \tag{7.4}$$
$$a_{51}c_1+a_{52}c_2+a_{53}c_3+a_{54}c_4+a_{55}c_5=\tfrac{1}{2}c_5^2 \tag{7.5}$$

$$a_{11}c_1^{\,2}+a_{12}c_2^{\,2}+a_{13}c_3^{\,2}+a_{14}c_4^{\,2}+a_{15}c_5^{\,2}=\tfrac{1}{3}c_1^3 \tag{8.1}$$
$$a_{21}c_1^{\,2}+a_{22}c_2^{\,2}+a_{23}c_3^{\,2}+a_{24}c_4^{\,2}+a_{25}c_5^{\,2}=\tfrac{1}{3}c_2^3 \tag{8.2}$$
$$a_{31}c_1^{\,2}+a_{32}c_2^{\,2}+a_{33}c_3^{\,2}+a_{34}c_4^{\,2}+a_{35}c_5^{\,2}=\tfrac{1}{3}c_3^3 \tag{8.3}$$
$$a_{41}c_1^{\,2}+a_{42}c_2^{\,2}+a_{43}c_3^{\,2}+a_{44}c_4^{\,2}+a_{45}c_5^{\,2}=\tfrac{1}{3}c_4^3 \tag{8.4}$$
$$a_{51}c_1^{\,2}+a_{52}c_2^{\,2}+a_{53}c_3^{\,2}+a_{54}c_4^{\,2}+a_{55}c_5^{\,2}=\tfrac{1}{3}c_5^3 \tag{8.5}$$

$$a_{11}c_1^{\,3}+a_{12}c_2^{\,3}+a_{13}c_3^{\,3}+a_{14}c_4^{\,3}+a_{15}c_5^{\,3}=\tfrac{1}{4}c_1^4 \tag{9.1}$$
$$a_{21}c_1^{\,3}+a_{22}c_2^{\,3}+a_{23}c_3^{\,3}+a_{24}c_4^{\,3}+a_{25}c_5^{\,3}=\tfrac{1}{4}c_2^4 \tag{9.2}$$
$$a_{31}c_1^{\,3}+a_{32}c_2^{\,3}+a_{33}c_3^{\,3}+a_{34}c_4^{\,3}+a_{35}c_5^{\,3}=\tfrac{1}{4}c_3^4 \tag{9.3}$$
$$a_{41}c_1^{\,3}+a_{42}c_2^{\,3}+a_{43}c_3^{\,3}+a_{44}c_4^{\,3}+a_{45}c_5^{\,3}=\tfrac{1}{4}c_4^4 \tag{9.4}$$
$$a_{51}c_1^{\,3}+a_{52}c_2^{\,3}+a_{53}c_3^{\,3}+a_{54}c_4^{\,3}+a_{55}c_5^{\,3}=\tfrac{1}{4}c_5^4 \tag{9.5}$$

$$a_{11}c_1^{\,4}+a_{12}c_2^{\,4}+a_{13}c_3^{\,4}+a_{14}c_4^{\,4}+a_{15}c_5^{\,4}=\tfrac{1}{5}c_1^5 \tag{10.1}$$
$$a_{21}c_1^{\,4}+a_{22}c_2^{\,4}+a_{23}c_3^{\,4}+a_{24}c_4^{\,4}+a_{25}c_5^{\,4}=\tfrac{1}{5}c_2^5 \tag{10.2}$$
$$a_{31}c_1^{\,4}+a_{32}c_2^{\,4}+a_{33}c_3^{\,4}+a_{34}c_4^{\,4}+a_{35}c_5^{\,4}=\tfrac{1}{5}c_3^5 \tag{10.3}$$
$$a_{41}c_1^{\,4}+a_{42}c_2^{\,4}+a_{43}c_3^{\,4}+a_{44}c_4^{\,4}+a_{45}c_5^{\,4}=\tfrac{1}{5}c_4^5 \tag{10.4}$$
$$a_{51}c_1^{\,4}+a_{52}c_2^{\,4}+a_{53}c_3^{\,4}+a_{54}c_4^{\,4}+a_{55}c_5^{\,4}=\tfrac{1}{4}c_5^5 \tag{10.5}$$

Then, from second equation in (5) we have the following systems

$$b_1+b_2+b_3+b_4+b_5=1 \tag{11.1}$$
$$b_1c_1+b_2c_2+b_3c_3+b_4c_4+b_5c_5=\tfrac{1}{2} \tag{11.2}$$
$$b_1c_1^{\,2}+b_2c_2^{\,2}+b_3c_3^{\,2}+b_4c_4^{\,2}+b_5c_5^{\,2}=\tfrac{1}{3} \tag{11.3}$$
$$b_1c_1^{\,3}+b_2c_2^{\,3}+b_3c_3^{\,3}+b_4c_4^{\,3}+b_5c_5^{\,3}=\tfrac{1}{4} \tag{11.4}$$
$$b_1c_1^{\,4}+b_2c_2^{\,4}+b_3c_3^{\,4}+b_4c_4^{\,4}+b_5c_5^{\,4}=\tfrac{1}{5} \tag{11.5}$$

Equations (11.1)-(11.5) are solved respect to with b_i $(i=1,2,3,4,5)$, we have the following form

$$b_1=\frac{\left\{\begin{array}{l}(12-15c_4+5(-3+4c_4)c_5-5c_3(3-4c_5+c_4(-4+6c_5))+\\ 5c_2(-3+c_4(4-6c_5)+4c_5+2c_3(2-3c_5+c_4(-3+6c_5))))\end{array}\right\}}{60(c_1-c_2)(c_1-c_3)(c_1-c_4)(c_1-c_5)} \tag{12.1}$$

$$b_2=\frac{\left\{\begin{array}{l}(5c_4(3-4c_5)+3(-4+5c_5)+5c_3(3-4c_5+c_4(-4+6c_5))+\\ 5c_1(3-4c_5+c_4(-4+6c_5)+c_3(-4+c_4(6-12c_5)+6c_5)))\end{array}\right\}}{60(c_1-c_2)(c_2-c_3)(c_2-c_4)(c_2-c_5)} \tag{12.2}$$

$$b_3=\frac{\left\{\begin{array}{l}12-15c_4+5(-3+4c_4)c_5-5c_2(3-4c_5+c_4(-4+6c_5))+\\ 5c_1(-3+c_4(4-6c_5)+4c_5+2c_2(2-3c_5+c_4(-3+6c_5))))\end{array}\right\}}{60(c_1-c_3)(c_2-c_3)(c_3-c_4)(c_3-c_5)} \tag{12.3}$$

$$b_4=\frac{\left\{\begin{array}{l}5c_3(3-4c_5)+3(-4+5c_5)+5c_2(3-4c_5+c_3(-4+6c_5))+\\ 5c_1(3-4c_5+c_3(-4+6c_5)+c_2(-4+c_3(6-12c_5)+6c_5)))\end{array}\right\}}{60(c_1-c_4)(-c_2+c_4)(-c_3+c_4)(c_4-c_5)} \tag{12.4}$$

$$b_5=\frac{\left\{\begin{array}{l}(12-15c_3+5(-3+4c_3)c_4-5c_2(3-4c_4+c_3(-4+6c_4))+\\ 5c_1(-3+c_3(4-6c_4)+4c_4+2c_2(2-3c_4+c_3(-3+6c_4))))\end{array}\right\}}{60(c_1-c_5)(c_2-c_5)(c_3-c_5)(c_4-c_5)} \tag{12.5}$$

Equations (6.1)-(10.5) solved respect to with a_{ij} $(i, j=1,2,3.4)$, we have the following form

$$a_{11}=\frac{\left\{\begin{array}{l}c_1(12c_4^4+60c_2c_3c_4c_5-15c_1^3(c_2+c_3+c_4+c_5)+20c_1^2(c_3c_4+(c_3+c_4)c_5+\\ c_2(c_3+c_4+c_5))-30c_1(c_3c_4c_5+c_2(c_3c_4+(c_3+c_4)c_5)))\end{array}\right\}}{60(c_1-c_2)(c_1-c_3)(c_1-c_4)(c_1-c_5)} \tag{13.1}$$

$$a_{12}=\frac{\left\{c_1^2(3c_1^3-30c_3c_4c_5-5c_1^2(c_3+c_4+c_5)+10c_1(c_4c_5+c_3(c_4+c_5))\right\}}{60(c_1-c_2)(c_2-c_3)(c_2-c_4)(c_2-c_5)} \tag{13.2}$$

$$a_{13}=\frac{\left\{c_1^2(-3c_1^3+30c_2c_4c_5+5c_1^2(c_2+c_4+c_5)-10c_1(c_4c_5+c_2(c_4+c_5)))\right\}}{60(c_1-c_3)(c_2-c_3)(c_3-c_4)(c_3-c_5)} \quad (13.3)$$

$$a_{14}=\frac{\left\{c_1^2(3c_1^3-30c_2c_3c_5-5c_1^2(c_2+c_3+c_5)+10c_1(c_3c_5+c_2(c_3+c_5)))\right\}}{60(c_1-c_4)(-c_2+c_4)(-c_3+c_4)(c_4+c_5)} \quad (13.4)$$

$$a_{15}=\frac{\left\{\begin{array}{l}12-15c_3+5(-3+4c_3)c_4-5c_2(3-4c_4+c_3(-4+6c_4))\\ +5c_1(-3+c_3(4-6c_4)+4c_4+2c_2(2-3c_4+c_3(-3+6c_4)))\end{array}\right\}}{60(c_1-c_5)(c_2+c_5)(c_3+c_5)(c_4+c_5)} \quad (13.5)$$

$$a_{21}=\frac{\left\{c_2^2((-3c_2^3+30c_5c_3c_4+5c_2^2(c_3+c_4+c_5)-10c_2(c_5c_4+c_3(c_5+c_4))))\right\}}{60(c_1-c_2)(c_1-c_3)(c_1-c_4)(c_1-c_5)} \quad (14.1)$$

$$a_{22}=\frac{\left\{\begin{array}{l}c_2(c_2(-12c_2^3+30c_3c_4c_5+15c_2^2(c_3+c_4+c_5)-20c_2(c_3c_4+(c_3+c_4)c_5))\\ +5c_1((3c_2^3-12c_3c_4c_5-4c_2^2(c_3+c_4+c_5)+6c_2(c_3c_4+(c_3+c_4)c_5))))\end{array}\right\}}{60(c_1-c_2)(c_2-c_3)(c_2-c_4)(c_2-c_5)} \quad (14.2)$$

$$a_{23}=\frac{\left\{c_2^2(5c_1(c_2^2+6c_4c_5-2c_2(c_4+c_5)))+c_2(-3c_2^2-10c_4c_5+5c_2(c_4+c_5))\right\}}{60(c_1-c_3)(c_2-c_3)(c_3-c_4)(c_3-c_5)} \quad (14.3)$$

$$a_{24}=\frac{\left\{c_2^2(3c_2^3+10c_3c_5-5c_2(c_3+c_5)-5c_1(c_2^2+6c_3c_5-2c_2(c_3+c_5)))\right\}}{60(c_1-c_4)(-c_2+c_4)(-c_3+c_4)(c_4+c_5)} \quad (14.4)$$

$$a_{25}=\frac{\left\{c_2^2(5c_1(c_2^2+6c_3c_4-2c_2(c_3+c_4))+c_2(-3c_2^2-10c_3c_4+5c_2(c_3+c_4)))\right\}}{60(c_1-c_5)(c_2+c_5)(c_3+c_5)(c_4+c_5)} \quad (14.5)$$

$$a_{31}=\frac{\left\{c_3^2(5c_2(c_3^2+6c_4c_5-2c_3(c_4+c_5))+c_3(-3c_3^2-10c_4c_5+5c_3(c_4+c_5)))\right\}}{60(c_1-c_2)(c_1-c_3)(c_1-c_4)(c_1-c_5)} \quad (15.1)$$

$$a_{32}=\frac{\left\{c_3^2(c_3(3c_3^2+10c_4c_5-5c_3(c_4+c_5))-5c_1(c_3^2+6c_4c_5-2c_3(c_4+c_5)))\right\}}{60(c_1-c_2)(c_2-c_3)(c_2-c_4)(c_2-c_5)} \quad (15.2)$$

$$a_{33}=\frac{\left\{\begin{array}{l}c_3(c_3(c_3(12c_3^2+20c_4c_5-15c_3(c_4+c_5))-5c_2(3c_3^2+6c_4c_5-4c_3(c_4+c_5)))+\\ 5c_1(2c_2(2c_3^2+6c_4c_5-3c_3(c_4+c_5))+c_3(-3c_3^2-6c_4c_5+4c_3(c_4+c_5))))\end{array}\right\}}{60(c_1-c_3)(c_2-c_3)(c_3-c_4)(c_3-c_5)} \quad (15.3)$$

$$a_{34}=\frac{\left\{c_3^2(c_3(-10c_1c_2+5(c_1+c_2)c_3-3c_3^2)+5(6c_1c_2-2(c_1+c_2)c_3+c_3^2)c_5)\right\}}{60(c_1-c_4)(-c_2+c_4)(-c_3+c_4)(c_4+c_5)} \quad (15.4)$$

$$a_{35}=\frac{\left\{c_3^2(c_3(-10c_1c_2+5(c_1+c_2)c_3-3c_3^2)+5(6c_1c_2-2(c_1+c_2)c_3+c_3^2)c_4))\right\}}{60(c_1-c_5)(c_2+c_5)(c_3+c_5)(c_4+c_5)} \quad (15.5)$$

$$a_{41}=\frac{\left\{((c_4^2(c_4(-10c_2c_3+5(c_2+c_3)c_4-3c_4^2)+5(6c_2c_3-2(c_2+c_3)c_4+c_4^2)c_5))\right\}}{60(c_1-c_2)(c_1-c_3)(c_1-c_4)(c_1-c_5)} \quad (16.1)$$

$$a_{42}=\frac{\left\{c_4^2(c_4(10c_1c_3-5(c_1+c_3)c_4+3c_4^2)-5(6c_1c_3-2(c_1+c_3)c_4+c_4^2)c_5)\right\}}{60(c_1-c_2)(c_2-c_3)(c_2-c_4)(c_2-c_5)} \quad (16.2)$$

$$a_{43}=\frac{\left\{c_4^2(c_4(10c_1c_2-5(c_1+c_2)c_4+3c_4^2)-5(6c_1c_2-2(c_1+c_2)c_4+c_4^2)c_5)\right\}}{60(c_1-c_3)(c_2-c_3)(c_3-c_4)(c_3-c_5)} \quad (16.3)$$

$$a_{44}=\frac{\left\{\begin{array}{l}c_4(c_4(30c_1c_2c_3-20(c_2c_3+c_1(c_2+c_3))c_4+15(c_1+c_2+c_3)c_4^2-12c_4^3)+\\ 5(-12c_1c_2c_3+6(c_2c_3+c_1(c_2+c_3))c_4-4(c_1+c_2+c_3)c_4^2+3c_4c_4^2)c_5)\end{array}\right\}}{60(c_1-c_4)(-c_2+c_4)(-c_3+c_4)(c_4+c_5)} \quad (16.4)$$

$$a_{45}=\frac{\left\{(c_4^2(30c_1c_2c_3-10(c_2c_3+c_1(c_2+c_3))c_4+5(c_1+c_2+c_3)c_4^2-3c_4c_4^2))\right\}}{60(c_1-c_5)(c_2+c_5)(c_3+c_5)(c_4+c_5)} \quad (16.5)$$

$$a_{51}=\frac{\left\{c_5^2(30c_2c_3c_4-10(c_3c_4+c_2(c_3+c_4))c_5+5(c_2+c_3+c_4)c_5^2-3c_5c_5^2)\right\}}{60(c_1-c_2)(c_1-c_3)(c_1-c_4)(c_1-c_5)} \quad (17.1)$$

$$a_{52}=\frac{\left\{c_5^2(-30c_1c_3c_4+10(c_3c_4+c_1(c_3+c_4))c_5-5(c_1+c_3+c_4)c_5^2+3c_5^3)\right\}}{60(c_1-c_2)(c_2-c_3)(c_2-c_4)(c_2-c_5)} \quad (17.2)$$

$$a_{53}=\frac{\left\{c_5^2(-30c_1c_2c_4+10(c_2c_4+c_1(c_2+c_4))c_5-5(c_1+c_2+c_4)c_5^2+3c_5^3)\right\}}{60(c_1-c_3)(c_2-c_3)(c_3-c_4)(c_3-c_5)} \quad (17.3)$$

$$a_{54}=\frac{\left\{c_5^2(-30c_1c_2c_3+10(c_2c_3+c_1(c_2+c_3))c_5-5(c_1+c_2+c_3)c_5^2+3c_5^3)\right\}}{60(c_1-c_4)(-c_2+c_4)(-c_3+c_4)(c_4+c_5)} \quad (17.4)$$

$$a_{55}=\frac{\left\{\begin{array}{l}c_5(-60c_1c_2c_3c_4+30(c_2c_3c_4+c_1(c_3c_4+c_2(c_3+c_4))c_5-20(c_3c_4+\\ c_2(c_3+c_4)+c_1(c_2+c_3+c_4))c_5^2+15(c_1+c_2+c_3+c_4)c_5^3-12c_5^4))\end{array}\right\}}{60(c_1-c_5)(c_2+c_5)(c_3+c_5)(c_4+c_5)} \quad (17.5)$$

In (12.1)-(17.5), if we set c_i $(i = 1, 2, 3, 4, 5)$ with a specific value, then we obtain b_i $(i = 1, 2, 3, 4, 5)$ and a_{ij} $(i, j = 1, 2, 3, 4, 5)$.

In IRK methods, c_i $(i = 1, 2, \dots, s)$ satisfies $\sum_{i=1}^{s} c_i = \frac{s}{2}$.

Therefore, $\sum_{i=1}^{s=5} c_i = 2\frac{1}{2}$ for 10th order IRK.

To make our computer simulation work efficient, we design an algorithm to find c_i $(i = 1, 2, 3, 4, 5)$ which must fulfill $\sum_{i=1}^{5} b_i = 1$ and $\sum_{i=1}^{5} c_i = 2\frac{1}{2}$. The algorithm is written in *pseudo code* as below :

Determining_The_Butcher_Coefficients_10th_order_IRK
{*Name of Algorithm***}**
{This algorithm is used to the determination simulation assess the Butcher's coefficients for the 10th order IRK method with the choice assess $0 < c_i < 1$, $i = 1, 2, 3, 4, 5.$ at random}

{Early Condition
The c_i $(i = 1, 2, 3, 4, 5)$ values selected random processing by using facility Randomize. Then, using the values, b_i $(i = 1, 2, 3, 4, 5)$, and a_{ij} $(i, j = 1, 2, \dots, 5)$ values will be obtained.
The values satisfy $\sum_{i=1}^{4} b_i = 1$,

$c_i = \sum_{j=1}^{5} a_{ij}$, and $\sum_{i=1}^{5} c_i = 2.5 \quad, i = 1..5.$ }

{Final Condition
c_i $(i = 1, 2, 3, 4, 5)$, b_i $(i = 1, 2, 3, 4, 5)$,and a_{ij} $(i, j = 1, 2, 3, 4, 5)$ values as Butcher's coefficients for the 10th order IRK method are obtained.}

DECLARATION

```
Label 10 I : integer
c1, c2, c3, c4, c5,b1,b2, b3,b4,b5,a11,a12,a13,a14
,a15,a21,a22,a23,a24,a25,a31,a32,a33,a34,a35,a41
,a42,a43,a44,a45,a51,a52,a53,a54,a55:real;

function f1(c1,c2,c3,c4,c5:real):real
begin
   f1 ← setting the rhs. Of equation 12.1 here
RETURN (f1)

function f2(c1,c2,c3,c4,c5:real):real
begin
   f2 ← setting the rhs. Of  equation 12.2 here
RETURN (f2)
function f3(c1,c2,c3,c4,c5:real):real
```

```
begin
    f3 ← setting the rhs. Of  equation 12.3 here
RETURN (f3)
function f4(c1,c2,c3,c4,c5:real):real
begin
   f4 ←  setting the rhs. Of  equation 12.4 here
RETURN (f4)
function f5(c1,c2,c3,c4,c5:real):real
begin
f5 ←  setting the rhs. Of  equation 12.5 here
RETURN (f5)
```

DESCRIPTION

```
  BEGIN
10:
    Randomize
    Repeat
     c1 ← Random  c2 ← Random
     c3 ← Random  c4 ← Random
     c5 ← Random

    until ((abs(c1+c2+c3+c4+c5 – 2.5) <= 0.0000001)  and
(c1>0)  and (c2>0)  and (c3>0)  and (c4>0) and (c5>0) and
(c1<>c2)  and (c2<>c3)  and (c3<>c4)  and (c4<>c5)  )

b1 ← f1(c1,c2,c3,c4,c5)    b2 ← f2(c1,c2,c3,c4,c5)
b3 ← f3(c1,c2,c3,c4,c5)    b4 ← f4(c1,c2,c3,c4,c5)
b5 ← f4(c1,c2,c3,c4,c5)
 if ( ((abs(b1+b2+b3+b4+b5  - 1 ) > 0.0000001) or (b1<0)
or (b2<0) or (b3<0) or (b4<0) or (b5<0)) ) then goto 10
else

BEGIN
       a11 ← setting the rhs. Of  equation 13.1 here
       a12 ← setting the rhs. Of  equation 13.2 here
       a13 ← setting the rhs. Of  equation 13.3 here
       a14 ← setting the rhs. Of  equation 13.4 here
       a15 ← setting the rhs. Of  equation 13.5 here
       a21 ← setting the rhs. Of  equation 14.1 here
       a22 ← setting the rhs. Of  equation 14.2 here
       a23 ← setting the rhs. Of  equation 14.3 here
       a24 ← setting the rhs. Of  equation 14.4 here
       a25 ← setting the rhs. Of  equation 14.5 here
       a31 ← setting the rhs. Of  equation 15.1 here
       a32 ← setting the rhs. Of  equation 15.2 here
       a33 ← setting the rhs. Of  equation 15.3 here
       a34 ← setting the rhs. Of  equation 15.4 here
       a35 ← setting the rhs. Of  equation 15.5 here
       a41 ← setting the rhs. Of  equation 16.1 here
       a42 ← setting the rhs. Of  equation 16.2 here
       a43 ← setting the rhs. Of  equation 16.3 here
       a44 ← setting the rhs. Of  equation 16.4 here
       a45 ← setting the rhs. Of  equation 16.5 here

       a51 ← setting the rhs. Of  equation 17.1 here
       a52 ← setting the rhs. Of  equation 17.2 here
       a53 ← setting the rhs. Of  equation 17.3 here
       a54 ← setting the rhs. Of  equation 17.4 here
       a55 ← setting the rhs. Of  equation 17.5 here
BEGIN
Write(c1, c2, c3, c4, c5,b1,b2, b3,b4,b5,
            ,a11,a12,a13,a14,a15,a21,a22,a23,a24,a25
            ,a31,a32,a33,a34,a35,a41,a42,a43,a44,a45
           ,a51,a52,a53,a54,a55)
   end
END.
```

Implementing the algorithm to TURBO PASCAL programming, we have a the 10th order of implicit Runge-Kutta methods that given in the form of a tableau containing their coefficients below

Table 1. The Butcher's coefficients for the 10th order IRK

0.00062327669	0.00063421625	0.00001105145	-0.00000999963	0.00000344788	0.00000094280
0.62262155069	0.09617192074	-0.08577858789	0.19691976268	0.44098804083	-0.02567958566
0.68561704247	0.09620099283	-0.05393518544	0.22869146117	0.44069256651	-0.02603279259
0.30589831341	0.10069140845	-0.57368596565	0.50477982176	0.31903127918	-0.04491823033
0.88523974386	0.09527844353	-0.20783782461	0.51158700582	0.44878101607	0.03743110305

III. Numerical Experiment

In this section, we present an application of our scheme to solve Henon-Heiles problem. The Hénon-Heiles problem is typically non integrable system and have simultaneously and quasi periodic solutions. The Hénon-Heiles system is defined by (see [8])

$$H = T + V, \quad T = \frac{1}{2}\left(p_1^2 + p_2^2\right), \quad V = \frac{1}{2}\left(q_1^2 + q_2^2\right) + \left(q_1^2 q_2\right) - \frac{1}{3}q_2^2 \qquad (18)$$

We apply the 10th order IRK method (**IRK$_{10}$**) to solve problem (18). The results of using **IRK$_{10}$** is displayed in the figure 1b. To show them, we used *MATHEMATICA*.

The figures displayed are *Poincaré* sections (i.e. sections of various orbits), for an energy *E* and time-steps ². To obtain our *Poincaré* sections, we use the idea of linear interpolation [10]. The *Poincaré* sections is the integration results as points of intersection of the flow with the $q_1 = 0$ -plane. Here we set 62 initial points.

In all graphs, the blue dots indicate that the points are moving up to $q_1 = 0$ -plane whereas the red dots indicate that the points are moving down the plane. Fig. 1.b shows that the scheme of 10th IRK method can be used to solve Hénon-Heiles problem (18).

In Fig. 1.a and Fig. 1.b, we show a poincaré section for Hénon-Heiles (18) using the 10th order IRK method whose tableau are given in Table.1 and a standard 4th order explicit Runge-Kutta methods, respectively. We set an energy E=1.25, ² = 0.01, and the number of iterations = 100,000 for producing the figures.

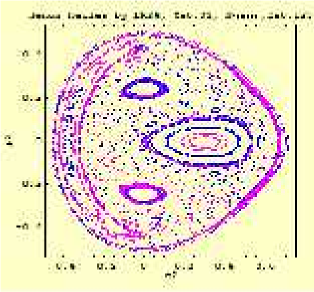

Fig. 1a. An Hénon-Heiles's *Poincaré* sections was produced by using a standard 4th order ERK method.

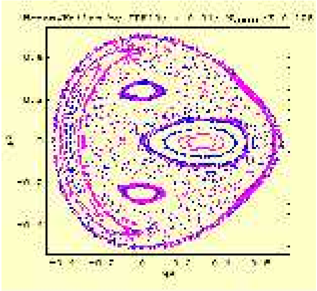

Fig. 1b. An Hénon-Heiles's *Poincaré* sections was produced by using an IRK_{10} method.

IV. CONCLUSION

Based on theoretical surveying, algorithm designing and implementing, and numerical problem solving before, we conclude that we can construct a scheme of 10th order of IRK methods via simulating to choose c_i $(i = 1..5)$ values that satisfy $\sum_{i=1}^{s=5} c_i = 2\frac{1}{2}, \sum_{i=1}^{5} b_i = 1$, and $c_i = \sum_{j=1}^{5} a_{ij}, i = 1..5\cdot$ Although, we have not exact values for c_i, b_i, and $a_{ij}, (i, j = 1, 2, 3.4)$, but our schemes can solve a problem of the first order of ordinary differential equation system like classical methods (Gauss-Legendre for example).

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