The Scheme of 10th Order Implicit Runge-Kutta Method to Solve the First Order of Initial Value Problems

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Abstract—To construct a scheme of implicit Runge-Kutta methods, there are a number of coefficients that must be determined and satisfying consistency properties and Butcher’s simplifying assumptions. In this paper we provide the numerical simulation technique to obtain a scheme of 10th order Implicit Runge-Kutta (IRK₁₀) method. For simulation process, we construct an algorithm to compute all the coefficients involved in the IRK₁₀ scheme. The algorithm is implemented in a language programming (Turbo Pascal) to obtain all the required coefficients in the scheme. To show that our scheme works correctly, we use the scheme to solve Hénon-Heiles system.

Keywords—ODEs, 10th order IRK method, numerical technique, Hénon-Heiles system

I. INTRODUCTION

Let a first order ordinary differential equation system (ODEs)

\[ y' = f(x, y(x)) \]

(1)

together with

\[ y(x₀) = y₀ \]

(2)

In (1), where the “prime” indicates differentiation with respect to \( x \), \( y \) is a \( D \)-dimensional vector \( \left( y \in \mathbb{R}^D \right) \), and \( f : \mathbb{R} \times \mathbb{R}^D \rightarrow \mathbb{R}^D \). The ODES (1)-(2) is the well-known first order of initial value problem. To solve problem (1)-(2) can be used analytical and/or numerical procedures. But, for solving the special problems (Hamiltonian and Divergen Free systems, for examples) and taking efficiency and effective calculations, some mathematicians recommended to use numerical approximations ([1-4,5]). One of numerical methods which can be used to solve (1-2) which is enough recognized and a lot of used is Runge-Kutta method.

Definition : Let \( b_i, a_{ij} \), and \( c_i (i, j = 1,2,\ldots,s) \) be real numbers. The method

\[ y_{n+1} = y_n + \tau \sum_{j=1}^{s} b_i \cdot f_i \]

(3)

\[ f_i = f \left( t_n + c_i \tau, y_n + \tau \sum_{j=1}^{s} a_{ij} \cdot f_j \right) \]

is called an \( s \)-stage Runge-Kutta method. When \( a_{ij} = 0 \) for \( i \leq j \) the method is called an explicit Runge-Kutta method. If \( a_{ij} = 0 \ i < j \) and least one \( a_{ii} \neq 0 \), the method is called a diagonal implicit Runge-Kutta method. If all of the diagonal elements are identical, \( a_{ij} = \gamma \) for \( i = 1,2,...,s \), the method is called a singly diagonal implicit Runge-Kutta method. In all other cases, the method is called an implicit Runge-Kutta method. The Runge-Kutta methods are often given in the form of a tableau containing their coefficients namely

\[
\begin{array}{c|cccc}
   & a_{11} & a_{12} & \cdots & a_{1s} \\
   c_1 & b_1 & b_2 & \cdots & b_s \\
   \vdots & \vdots & \vdots & \ddots & \vdots \\
   c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\end{array}
\]

where \( c_i = \sum_{j=1}^{s} a_{ij}; i = 1,2,\ldots,s \).

Consistency in the Runge-Kutta methods is investigated by using a Taylor series expansion. Any \( s \)-stage Runge-Kutta process of the form (3) or (4) is consistent if \( \sum_{i=1}^{s} b_i = 1 \).

Butcher in [6] states that this condition is necessary and sufficient condition for the local truncation error of the method to have asymptotic behavior \( O(\tau) \).

There are some advantages to use Implicit Runge-Kutta (IRK) methods (see [6-8]). (1) IRK methods usually are required for systems whose solutions contain rapidly decaying components (see [6]); (2) Some methods may also be used in preserving the symplectic structure of Hamiltonian systems; (3) IRK methods (Gauss quadratures) have a number of big potency for the computing of integrate the geometry; (4) high order integrator to be used by the reason of doubled accuracy is recommended; (5) to evaluate the vector field "costly", all stage-s in IRK can be evaluated by parallel and; (6) IRK can be solve the ordinary differential equation (ODE) or system (ODEs) in general.

To construct a scheme of IRK methods, Butcher discovered the existence of \( s \)-stage methods of order \( 2s \), for all \( s \). He used simplifying assumptions to find these methods. The simplifying assumptions are

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\[ A(\xi) = \sum_{j=1}^{q} a_j c_j^{s-1} = \frac{1}{q} c_q^s \quad i=1,2,\ldots,s \quad q=1,2,\ldots,\xi \]

\[ B(p) = \sum_{j=1}^{q} b_j c_j^{s-1} = \frac{1}{q} \quad q=1,2,\ldots,p \]  

\[ C(p) = \sum_{j=1}^{q} b_j c_j^{s-1} a_{ij} = \frac{b_j}{q} (1-c_j^s) \quad j=1,2,\ldots,s \quad q=1,2,\ldots,p \]


In analytical procedure, we can use the idea of collocation to derive IRK methods ([4],[8]). Unfortunately, for the stage of \( s \geq 3 \), constructing an integrator IRK using analytical procedure will be difficult because there are some values of \( b_i, c_i \), and \( a_{ij} \) \((i, j=1,2,\ldots,s)\) must be determined.

Therefore, the numerical technique or procedure is an alternative choice to use. In this procedure, we can use computer to determine the values of \( c_i \) \((i=1,2,\ldots,s)\) , \( b_i \) and \( a_{ij} \) \((i=1,2,\ldots,s)\).

Notice that one important property of IRK method is symplecteness. An IRK method (4) is symplectic if it satisfy the following condition \([9]\)

\[ b_i b_{j} - b_{i} a_{ij} - b_{j} a_{ji} = 0 \quad i,j \in \{1,2,\ldots,s\} \]

II. RESULT AND DISCUSSION

In this section, we describe how to get a class of 10th order IRK method using computer simulation to obtain all Butcher’s coefficient values (4) based on Butcher’s simplifying assumptions (5).

Setting \( k = i = j = s = 1,2,3,4,5 \) into first equation in (5), we have the following systems

\[ a_{11} + a_{12} + a_{13} + a_{14} + a_{15} = c_1 \]  

\[ a_{21} + a_{22} + a_{23} + a_{24} + a_{25} = c_2 \]  

\[ a_{31} + a_{32} + a_{33} + a_{34} + a_{35} = c_3 \]  

\[ a_{41} + a_{42} + a_{43} + a_{44} + a_{45} = c_4 \]  

\[ a_{51} + a_{52} + a_{53} + a_{54} + a_{55} = c_5 \]  

\[ a_{11} c_1 + a_{12} c_2 + a_{13} c_3 + a_{14} c_4 + a_{15} c_5 = \frac{1}{2} c_1^2 \]  

\[ a_{21} c_1 + a_{22} c_2 + a_{23} c_3 + a_{24} c_4 + a_{25} c_5 = \frac{1}{2} c_2^2 \]  

\[ a_{31} c_1 + a_{32} c_2 + a_{33} c_3 + a_{34} c_4 + a_{35} c_5 = \frac{1}{2} c_3^2 \]  

\[ a_{41} c_1 + a_{42} c_2 + a_{43} c_3 + a_{44} c_4 + a_{45} c_5 = \frac{1}{2} c_4^2 \]  

\[ a_{51} c_1 + a_{52} c_2 + a_{53} c_3 + a_{54} c_4 + a_{55} c_5 = \frac{1}{2} c_5^2 \]  

\[ a_{11} c_1^2 + a_{12} c_2^2 + a_{13} c_3^2 + a_{14} c_4^2 + a_{15} c_5^2 = \frac{1}{4} c_1^3 \]  

\[ a_{21} c_1^2 + a_{22} c_2^2 + a_{23} c_3^2 + a_{24} c_4^2 + a_{25} c_5^2 = \frac{1}{4} c_2^3 \]  

\[ a_{31} c_1^2 + a_{32} c_2^2 + a_{33} c_3^2 + a_{34} c_4^2 + a_{35} c_5^2 = \frac{1}{4} c_3^3 \]  

\[ a_{41} c_1^2 + a_{42} c_2^2 + a_{43} c_3^2 + a_{44} c_4^2 + a_{45} c_5^2 = \frac{1}{4} c_4^3 \]  

\[ a_{51} c_1^2 + a_{52} c_2^2 + a_{53} c_3^2 + a_{54} c_4^2 + a_{55} c_5^2 = \frac{1}{4} c_5^3 \]  

\[ a_{11} c_1^3 + a_{12} c_2^3 + a_{13} c_3^3 + a_{14} c_4^3 + a_{15} c_5^3 = \frac{1}{8} c_1^4 \]  

\[ a_{21} c_1^3 + a_{22} c_2^3 + a_{23} c_3^3 + a_{24} c_4^3 + a_{25} c_5^3 = \frac{1}{8} c_2^4 \]  

\[ a_{31} c_1^3 + a_{32} c_2^3 + a_{33} c_3^3 + a_{34} c_4^3 + a_{35} c_5^3 = \frac{1}{8} c_3^4 \]  

\[ a_{41} c_1^3 + a_{42} c_2^3 + a_{43} c_3^3 + a_{44} c_4^3 + a_{45} c_5^3 = \frac{1}{8} c_4^4 \]  

\[ a_{51} c_1^3 + a_{52} c_2^3 + a_{53} c_3^3 + a_{54} c_4^3 + a_{55} c_5^3 = \frac{1}{8} c_5^4 \]  

\[ a_{11} c_1^4 + a_{12} c_2^4 + a_{13} c_3^4 + a_{14} c_4^4 + a_{15} c_5^4 = \frac{1}{16} c_1^5 \]  

\[ a_{21} c_1^4 + a_{22} c_2^4 + a_{23} c_3^4 + a_{24} c_4^4 + a_{25} c_5^4 = \frac{1}{16} c_2^5 \]  

\[ a_{31} c_1^4 + a_{32} c_2^4 + a_{33} c_3^4 + a_{34} c_4^4 + a_{35} c_5^4 = \frac{1}{16} c_3^5 \]  

\[ a_{41} c_1^4 + a_{42} c_2^4 + a_{43} c_3^4 + a_{44} c_4^4 + a_{45} c_5^4 = \frac{1}{16} c_4^5 \]  

\[ a_{51} c_1^4 + a_{52} c_2^4 + a_{53} c_3^4 + a_{54} c_4^4 + a_{55} c_5^4 = \frac{1}{16} c_5^5 \]  

Then, from second equation in (5) we have the following systems

\[ b_i + b_2 + b_3 + b_4 + b_5 = 1 \]  

\[ b_{c_1} + b_{c_2} + b_{c_3} + b_{c_4} + b_{c_5} = \frac{1}{2} \]  

\[ b_{c_1}^2 + b_{c_2}^2 + b_{c_3}^2 + b_{c_4}^2 + b_{c_5}^2 = \frac{1}{4} \]  

\[ b_{c_1}^3 + b_{c_2}^3 + b_{c_3}^3 + b_{c_4}^3 + b_{c_5}^3 = \frac{1}{8} \]  

\[ b_{c_1}^4 + b_{c_2}^4 + b_{c_3}^4 + b_{c_4}^4 + b_{c_5}^4 = \frac{1}{16} \]  

Equations (11.1)-(11.5) are solved respect to with \( b_i \) \((i=1,2,3,4,5)\) , we have the following form

(12.1)  

(12.2)  

(12.3)  

(12.4)  

(12.5)  

Equations (6.1)-(10.5) solved respect to with \( a_{ij} \) \((i=1,2,3,4,5)\) , we have the following form

(13.1)  

(13.2)
The values satisfy 
\[ \sum_{i=1}^{5} a_i = 1 \]
(15.4)
The values satisfy \( \sum_{i=1}^{5} b_i = 1 \).
(15.5)
\[ c_i = \sum_{j=1}^{5} a_{ij}, \quad \sum_{i=1}^{5} c_i = 2.5 \quad i = 1, \ldots, 5. \]
(16.1)
\[ b_j (i = 1, 2, 3, 4, 5), \quad a_{ij} (i, j = 1, 2, 3, 4, 5) \]
values as Butcher’s coefficients for the 10th order IRK method are obtained.
(16.2)
\[ \text{DECLARATION} \]
Label 10 : integer
c1, c2, c3, c4, c5, b1, b2, b3, b4, b5, a11, a12, a13, a14,
a15, a21, a22, a23, a24, a25, a31, a32, a33, a34, a35, a41,
a42, a43, a44, a45, a51, a52, a53, a54, a55: real;
(16.4)
\[ \text{function } f1(c1,c2,c3,c4,c5:real):real\r
\text{ begin} \] 
f1 ← setting the rhs. Of equation 12.1 here 
\text{RETURN (f1)}
(17.1)
\[ \text{function } f2(c1,c2,c3,c4,c5:real):real\r
\text{ begin} \] 
f2 ← setting the rhs. Of equation 12.2 here 
\text{RETURN (f2)}
(17.2)
\[ \text{function } f3(c1,c2,c3,c4,c5:real):real; \] 
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begin
  f3 ← setting the rhs. Of equation 12.3 here
RETURN (f3)
function f4(c1,c2,c3,c4,c5:real):real;
begin
  f4 ← setting the rhs. Of equation 12.4 here
RETURN (f4)
function f5(c1,c2,c3,c4,c5:real):real;
begin
  f5 ← setting the rhs. Of equation 12.5 here
RETURN (f5)

DESCRIPTION
BEGIN
10:
  Randomize
Repeat
  c1 ← Random  c2 ← Random
  c3 ← Random  c4 ← Random
  c5 ← Random
  until ((abs(c1+c2+c3+c4+c5) - 2.5) <= 0.0000001) and
         (c1>0) and (c2>0) and (c3>0) and (c4>0) and (c5>0) and
         (c1<>c2) and (c2<>c3) and (c3<>c4) and (c4<>c5)

b1 ← f1(c1,c2,c3,c4,c5)  b2 ← f2(c1,c2,c3,c4,c5)
b3 ← f3(c1,c2,c3,c4,c5)  b4 ← f4(c1,c2,c3,c4,c5)
b5 ← f5(c1,c2,c3,c4,c5)
if ( ((abs(b1+b2+b3+b4+b5) - 0.0000001) or (b1<0)
     or (b2<0) or (b3<0) or (b4<0) or (b5<0) )
     ) then goto 10
else
  
BEGIN
a11 ← setting the rhs. Of equation 13.1 here
a12 ← setting the rhs. Of equation 13.2 here
a13 ← setting the rhs. Of equation 13.3 here
a14 ← setting the rhs. Of equation 13.4 here
a15 ← setting the rhs. Of equation 13.5 here
a21 ← setting the rhs. Of equation 14.1 here
a22 ← setting the rhs. Of equation 14.2 here
a23 ← setting the rhs. Of equation 14.3 here
a24 ← setting the rhs. Of equation 14.4 here
a25 ← setting the rhs. Of equation 14.5 here
a31 ← setting the rhs. Of equation 15.1 here
a32 ← setting the rhs. Of equation 15.2 here
a33 ← setting the rhs. Of equation 15.3 here
a34 ← setting the rhs. Of equation 15.4 here
a35 ← setting the rhs. Of equation 15.5 here
a41 ← setting the rhs. Of equation 16.1 here
a42 ← setting the rhs. Of equation 16.2 here
a43 ← setting the rhs. Of equation 16.3 here
a44 ← setting the rhs. Of equation 16.4 here
a45 ← setting the rhs. Of equation 16.5 here
a51 ← setting the rhs. Of equation 17.1 here
a52 ← setting the rhs. Of equation 17.2 here
a53 ← setting the rhs. Of equation 17.3 here
a54 ← setting the rhs. Of equation 17.4 here
a55 ← setting the rhs. Of equation 17.5 here

III. NUMERICAL EXPERIMENT

In this section, we present an application of our scheme to solve Henon-Heiles problem. The Hénon-Heiles problem is typically non integrable system and have simultaneously and quasi periodic solutions. The Hénon-Heiles system is defined by (see [8])

\[ H= T + V , \quad T= \frac{1}{2} \left( p_i^2 + q_i^2 \right) , \quad V= \frac{1}{2} \left( q_i^2 + q_i' \right) \] + \left( q_i q_i' \right) \frac{1}{3} q_i'^3 \] (18)

We apply the 10th order IRK method (IRK10) to solve problem (18). The results of using IRK10 is displayed in the figure 1b. To show them, we used MATHEMATICA.

The figures displayed are Poincaré sections (i.e. sections of various orbits), for an energy E and time-steps τ. To obtain our Poincaré sections, we use the idea of linear interpolation [10]. The Poincaré sections is the integration results as points of intersection of the flow with the q_i = 0 -plane. Here we set 62 initial points.

In all graphs, the blue dots indicate that the points are moving up to q_i = 0-plane whereas the red dots indicate that the points are moving down the plane. Fig. 1.b shows that the scheme of 10th IRK method can be used to solve Hénon-Heiles problem (18).

In Fig. 1.a and Fig. 1.b, we show a poincaré section for Hénon-Heiles (18) using the 10th order IRK method whose tableau are given in Table.1 and a standard 4th order explicit Runge-Kutta methods, respectively. We set an energy E=1.25, τ = 0.01, and the number of iterations = 100,000 for producing the figures.
IV. CONCLUSION

Based on theoretical surveying, algorithm designing and implementing, and numerical problem solving before, we conclude that we can construct a scheme of 10th order of IRK methods via simulating to choose \( c_i \) \((i = 1..5)\) values that satisfy \( \sum_{i=1}^{5} c_i = 2 \sum_{j=1}^{5} b_j = 1 \) and \( c_i = \sum_{j=1}^{5} a_{ij}, \) \( i = 1..5 \).

Although, we have not exact values for \( c_i, b_j, \) and \( a_{ij}, \) \((i, j = 1, 2, 3, 4)\), but our schemes can solve a problem of the first order of ordinary differential equation system like classical methods (Gauss-Legendre for example).

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